Proof of a Conjecture on Immanants of the Jacobi-Trudi Matrix

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ABSTRACT

We prove a conjecture on characters of S_n which implies another conjecture (both due to Goulden and Jackson) that all immanants of the Jacobi-Trudi matrix $H(\nu,\mu)=(h_{(\nu_i-i)-(\mu_j-j)})_{i,j=1,\ldots,n}$ have nonnegative coefficients.

1. INTRODUCTION: IMMANANTS AND JACOBI-TRUDI MATRICES

Let A be an $n\times n$ matrix, and let χ^λ denote the irreducible character of S_n associated with the partition $\lambda\vdash n$. The function

$$\operatorname{Imm}_\lambda A=\sum_{\sigma\in S_n}\chi^\lambda(\sigma)\prod_{i=1}^{n}a_{i\sigma(i)}$$

is called the *immanant of A with respect to* λ. By taking χ^λ to be the sign character and the trivial character, respectively, one obtains the determinant and permanent of A as special cases. The theory of immanants is minuscule by comparison with the rich theory of determinants; nonetheless a few substantial facts are known. An important example is Schur's inequality [15]

$$\frac{1}{\chi^\lambda(I)}\operatorname{Imm}_\lambda A\geqslant\det A\geqslant 0,$$

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which holds when A is a positive semidefinite Hermitian matrix. An outstanding open question [9] is the corresponding inequality for permanents:

$$\operatorname{per} A \geq \frac{1}{\chi^\lambda(I)} \operatorname{Imm}_\lambda A.$$

A good survey of this problem and related results can be found in [14].

The word "immanant" apparently appears first in [11].[1] In [10] Littlewood uses immanants to introduce the Schur symmetric functions s_λ for $\lambda \vdash n$, defining them by the formula

$$n! s_\lambda = \operatorname{Imm}_\lambda \begin{pmatrix} p_1 & 1 & 0 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & 0 & \cdots & 0 \\ p_3 & p_2 & p_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.$$

The matrix is $n \times n$, and p_i denotes the ith power-sum symmetric function in the variables $x_1, x_2, x_3, \ldots$.

In this paper we will be interested in immanants of a special matrix $H(\nu, \mu)$ defined as follows. Suppose that $\nu = \{\nu_1, \nu_2, \ldots, \nu_m\}$ and $\mu = \{\mu_1, \mu_2, \ldots, \mu_m\}$ are partitions, with $\nu_i \geqslant \mu_i \geqslant 0$ for $i = 1, \ldots, m$. Define

$$H(\nu, \mu) = \left(h_{(\nu_i - i) - (\mu_j - j)}\right)_{i,j=1,\ldots,n},$$

where $h_k = h_k(x_1, x_2, \ldots)$ denotes the kth homogeneous symmetric function (the sum of all monomials of degree k in the variables x_i). By convention, $h_0 = 1$ and $h_k = 0$ if $k < 0$. The matrix $H(\nu, \mu)$ is called the *Jacobi-Trudi matrix*, after Jacobi and Trudi, who independently proved that

$$\det H(\nu, \emptyset) = s_\nu \tag{1}$$

and more generally

$$\det H(\nu, \mu) = s_{\nu/\mu}, \tag{2}$$

where $s_{\nu/\mu}$ denotes a skew Schur function (and $\emptyset$ denotes the empty

[1] The authors of that paper credit A. Forsythe with inventing the name.

partition). The latter is a well-defined linear combination of ordinary Schur functions, with nonnegative integer coefficients. (See [12] or [16] for more details about these and other facts about symmetric functions.)

It is well known that the Schur functions s_ν (and hence $s_{\nu/\mu}$) have nonnegative expansions in monomials. The coefficients are the Kostka numbers, which count column strict plane partitions of a given shape and content. Goulden and Jackson conjectured in [6] that a similar result holds for arbitrary immanants:

CONJECTURE 1.1 [6]. Let ν and μ be partitions with at most m parts, with $\nu_i \geq \mu_i$, $i = 1, \ldots, m$, and let λ be a partition of m. Write

$$\mathrm{Imm}_\lambda\, H(\nu, \mu) = \sum_\theta c(\lambda, \nu, \mu, \theta) m_\theta,$$

where m_θ denotes the monomial symmetric function. Then $c(\lambda, \nu, \mu, \theta) \geqslant 0$ for all θ.

A clever combinatorial argument in [6] reduces Conjecture 1.1 to the following purely character-theoretic statement:

CONJECTURE 1.2 [6]. Let $J_1, J_2, \ldots, J_P$ denote a collection of subintervals of $[\mathbf{n}] = \{1, 2, \ldots, n\}$. For an interval J let S_J denote the sum (in the group algebra of S_n) of all permutations of the elements in J. Then for every irreducible character χ^λ of S_n,

$$\chi^\lambda(S_{J_1} S_{J_2} \cdots S_{J_P}) \geqslant 0.$$

In this paper we settle both conjectures by proving the following stronger result:

THEOREM 1.3. *Let $J \subseteq [\mathbf{n}]$ be an interval, and let $\rho_\lambda(S_J)$ denote the matrix representing S_J in Young's seminormal representation of S_n, indexed by λ. Then $\rho_\lambda(S_J)$ is a matrix in which every entry is nonnegative.*

We prove Theorem 1.3 in Section 4 by deriving explicit expressions for the matrix entries of $\rho_\lambda(S_J)$, which turn out to have interesting combinatorial structure. The representations ρ_λ will be described in detail in Section 3. Section 2 gives (for completeness) a sketch of Goulden and Jackson's argument [6] reducing Conjecture 1.1 to Conjecture 1.2.

We note that Stembridge [17] has made the following stronger conjecture, which remains open[2] and apparently beyond the reach of these methods:

CONJECTURE 1.4 [17]. With notation as above, write

$$\operatorname{Imm}_\lambda H(\nu,\mu) = \sum_\theta \bar{c}(\lambda,\nu,\mu,\theta) s_\theta,$$

where s_θ denotes the Schur symmetric function. Then $\bar{c}(\lambda,\nu,\mu,\theta) \geqslant 0$ for all θ.

ILLUSTRATION. If $m = 3$, $\nu = \{2,1,1\}$, and $\nu = \emptyset$, then

$$H(\nu,\mu) = \begin{pmatrix} h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{pmatrix}$$

and

$$\begin{aligned}
\operatorname{Imm}_{1^3} H(\nu,\mu) &= \det H(\nu,\mu) \\
&= h_1^2 h_2 + h_4 - h_2^2 - h_1 h_3 \\
&= m_{2,1,1} + 3m_{1,1,1,1} \\
&= s_{2,1,1}, \\
\operatorname{Imm}_3 H(\nu,\mu) &= \operatorname{per} H(\nu,\mu) \\
&= h_1^2 h_2 + h_4 + h_2^2 + h_1 h_3 \\
&= 4m_4 + 8m_{3,1} + 10m_{2,2} + 15m_{2,1,1} + 23m_{1,1,1,1} \\
&= 4s_4 + 4s_{3,1} + 2s_{2,2} + s_{2,1,1}.
\end{aligned}$$

There is one additional character χ of S_3, of degree 2, corresponding to the

[2]Since this paper was written, Conjecture 1.4 has been solved by Mark Haiman [7].

partition $\lambda = \{2,1\}$. For this character we have $\chi(I) = 2$, $\chi(\tau) = 0$ for a transposition τ, and $\chi(\theta) = -1$ for a 3-cycle θ. Thus

$$\begin{aligned}\mathrm{Imm}_{2,1} H(\nu,\mu) &= 2h_1^2 h_2 - h_4\\ &= m_4 + 5m_{3,1} + 7m_{2,2} + 13m_{2,1,1} + 23m_{1,1,1,1}\\ &= s_4 + 4s_{3,1} + 2s_{2,2} + 2s_{2,1,1}.\end{aligned}$$

2. LATTICE PATHS

The goal of this section will be to interpret the terms

$$\prod_{i=1}^{n} h_{(\nu_i - i) - (\mu_{\sigma(i)} - \sigma(i))} \tag{3}$$

arising in the expansion of $\mathrm{Imm}_\lambda H(\nu,\mu)$ as generating functions for certain collections of lattice paths. This approach was used by Gessel and Viennot ([3]; see also [4]) to give a combinatorial interpretation of the Jacobi-Trudi identity

$$\det H(\nu,\mu) = s_{\nu/\mu}.$$

Goulden and Jackson showed how to apply the same ideas to arbitrary immanants. The exposition in this section follows [6].

The key idea is that h_k can be interpreted as a generating function for (northeast) lattice paths which start at $(0,1)$ and end at (k,∞) (i.e. go to ∞ along the line $x = k$), weighting horizontal steps at height i by x_i and weighting a path by the product of the weights of its horizontal steps. Clearly h_k also enumerates paths from $(a,1)$ to $(a+k,\infty)$, for any integer a. Thus the product (3) may be viewed as a generating function for families of n paths, connecting $(\mu_{\sigma(i)} - \sigma(i), 1)$ to $(\nu_i - i, \infty)$, for $i = 1,\dots,n$, with weights as defined above.

Define $P_i = (\mu_i - i, 1)$ and $Q_i = (\nu_i - i, \infty)$, for $i = 1,\dots,n$. Let F be a family of paths connecting the points P_i to some permutation of the Q_i, and let $\alpha = \alpha_F$ denote the multiset of edges (steps) in the union of all of the paths in F. We call α_F the *skeleton* of F.[3] Let x^α denote the monomial

[3]The term *skeleton* was suggested by John Stembridge.

obtained by taking the product of weights in all of the horizontal steps. Then

$$\begin{aligned}\mathrm{Imm}_\lambda H(\nu,\mu) &= \sum_\sigma \chi^\lambda(\sigma) \prod_{i=1}^n h_{(\nu_i - i)-(\mu_{\sigma(i)}-\sigma(i))}\\ &= \sum_\sigma \chi^\lambda(\sigma) \sum_F x^{\alpha_F},\end{aligned}$$

where the inner sum is over families F corresponding to a given permutation σ. Reversing the order of summation and collecting monomials corresponding to a fixed skeleton α, one obtains

$$\mathrm{Imm}_\lambda H(\nu,\mu) = \sum_\alpha x^\alpha \sum_F \chi^\lambda(\sigma_F), \tag{4}$$

where the inner sum is over all families of paths F having skeleton α, and σ_F denotes the permutation of endpoints induced by F. The key observation in [6] is that, for a fixed skeleton α, the element

$$\sum_F \sigma_F \in \mathrm{CS}_n \tag{5}$$

determined by the inner sum in (4) has exactly the form

$$kS_{J_1}S_{J_2}\cdots S_{J_P}, \tag{6}$$

where $J_1, J_2, \ldots, J_P$ are intervals of [n] and $k \geqslant 0$.

To see this, temporarily assume that the multiple edges occurring in α are distinguishable, so that switching two of them results in a different family F. Assume also that the paths extend vertically downward from $(P_i, 1)$ to $(P_i, -\infty)$, for each i. Next imagine a continuous arc stretching from $(-\infty, 0)$ to from $(+\infty, 0)$, passing below $P_1, P_2, \ldots, P_n$. Deform the arc continuously in a vertical direction, crossing one intersection point at a time, until only $Q_1, Q_2, \ldots, Q_n$ remain above. At each stage of the deformation, the order in which the paths in a fixed family F cross the arc determines a permutation $\sigma_F \in S_n$. The element determined by (5) is obtain by summing the permutations σ_F over all F, at the final stage. It is clear that in crossing an intersection point V at an intermediate stage, the sum of all permutations induced by path families below the arc is multiplied by S_J, where J is determined by the set of arcs leaving V. It follows from the planarity of the skeleton that J is an interval.

Finally, observe that the permutations are overcounted by a factor of $E = \prod e_i!$, where e_i denotes the multiplicity of the ith edge. Taking $k = 1/E$ completes the proof of the claim made above.

ILLUSTRATION. If $m = 3$, $\nu = \{2,1,1\}$, and $\mu = \emptyset$, then $P_1 = (-1, 1,)$, $P_2 = (-2, 1)$, $P_3 = (-3, 1)$, and $Q_1 = (1, \infty)$, $Q_2 = (-1, \infty)$, $Q_3 = (-2, \infty)$. A possible skeleton α is shown in Figure 1. The weight of α is $x^\alpha = x_2^2 x_3^2$. The symmetric groups determined by the three intersection points are $S_{\{2,3\}}$, $S_{\{1,2\}}$, and $S_{\{1,2\}}$, and there is one double edge. Hence the coefficient of x^α in the expansion (4) is

$$\tfrac{1}{2}\chi^\lambda(S_{\{2,3\}} S_{\{1,2\}} S_{\{1,2\}}) = \chi^\lambda(I + (1,2) + (2,3) + (1,2,3)),$$

which is equal to 0, 4, or 1, according as $\lambda = \{1^3\}$, $\{3\}$, or $\{2,1\}$.

REMARK. If $\lambda = \{1^n\}$, so that χ^λ is the sign character, it is easy to see that $\chi^\lambda(S_{J_1}, S_{J_2} \cdots S_{J_P}) = 0$ if some $|J_i| > 0$. Hence the contribution to $\mathrm{Imm}_\lambda H$ by *intersecting* families of lattice paths disappears. This is another (essentially equivalent) way of expressing the cancellation step in the Gessel-Viennot proof of the Jacobi-Trudi identity.

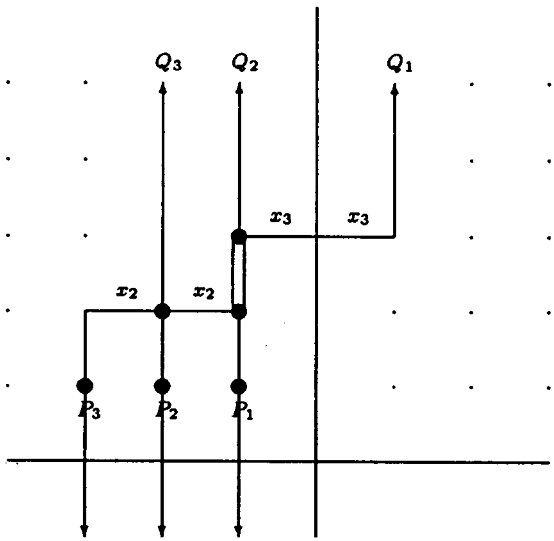

FIG. 1. Example of a skeleton; $\nu = \{2,1,1\}$ and $\mu = \emptyset$.

3. YOUNG'S SEMINORMAL FORM

In this section we give the construction of Young's seminormal representations ρ_λ of S_n. For a more thorough treatment we refer the reader to [8]. If $\lambda \vdash n$, the degree of ρ_λ is equal to f_λ, the number of standard Young tableaux of shape λ, and we may take these tableaux as basis vectors for the representation space. It is convenient to index tableaux in the so-called *last-letter order*. In this order, tableaux in which the last letter is lower come later. This principle is applied successively to $n, n-1, n-2, \ldots$ to determine a total order. Let T_i denote the ith standard tableau of shape λ in this order. For example, if $\lambda = \{3,2\}$, the tableaux are

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}}$$

The construction begins by defining matrices $\rho_\lambda(\tau_k)$ for the adjacent transpositions $\tau_k = (k, k+1)$, $k = 1, 2, \ldots, n-1$. Then $\rho_\lambda(\sigma)$ is computed for arbitrary σ by taking products.

DEFINITION 3.1. Let p and q be elements which appear in a tableau T. Let $\delta(p,q)$ denote the (signed) distance from p to q in T. That is, if p lies in row r and column c, and q lies in row r' and column c', then $\delta(p,q) = (c' - r') - (c - r)$. In the literature, $\delta(p,q)$ is called the *axial distance* from p to q in T.

DEFINITION 3.2 [Definition of matrix $\hat{\tau}_k = \rho_\lambda(\tau_k)$ when $\tau_k = (k, k+1)$]. Let $\delta_i = \delta(k, k+1)$ be the axial distance from k to $k+1$ in the ith tableau T_i, as defined above.

(1) Diagonal entries: $\rho_\lambda(\tau_k)|_{i,i} = 1/\delta_i$ for all i.

(2) Off-diagonal entries:

(a) If $i \neq j$ and $T_j \neq \tau_k T_i$, then $\rho_\lambda(\tau_k)|_{i,j} = 0$.
(b) Otherwise, if $T_j = \tau_k T_i$, then

$$\rho_\lambda(\tau_k)|_{i,j} = \begin{cases} 1 - 1/\delta_i^2 & \text{if } \ i < j, \\ 1 & \text{if } \ i > j. \end{cases}$$

According to this definition, $\rho_\lambda(\tau_k)$ is block diagonal, with blocks of size either 1 or 2. (The blocks need not be formed from consecutive rows and columns.) Rows i and j determine a 2×2 block if k and $k+1$ lie in distinct rows and columns in T_i and T_j, and $T_j = \tau_k T_i$. If k and $k+1$ lie in the same row or column of T_i, then T_i determines a 1×1 block, and the diagonal entry is ± 1.

ILLUSTRATION. If $\lambda = \{3,2\}$ and $\tau_2 = (2,3)$, then

$$\hat{\tau} = \rho_\lambda(\tau_2) = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. PROOF OF THEOREM 1.3

We will prove Theorem 1.3 by obtaining explicit expressions for the entries of $\rho_\lambda(S_J)$.

DEFINITION 4.1. If T is a standard tableau, a pair of entries $\{p, q\}$ is an *inversion* in T if $p < q$ and q appears in a lower row than p. Otherwise $\{p, q\}$ is a *noninversion*. Inversions and noninversions are called *strict* if p and q lie in different rows and columns.

The exact description of $\rho_\lambda(S_J)$ is given by the following theorem.

THEOREM 4.2. *Let $J \subseteq [\mathrm{n}]$ be an interval. Then*

(1) $\rho_\lambda(S_J)|_{i,j} = 0$ *unless* $T_j = \sigma T_i$ *for some* $\sigma \in S_J$.
(2) $\rho_\lambda(S_J)|_{i,j} = 0$ *unless the elements of J lie in distinct columns of* T_i.
(3) *If conditions* (1) *and* (2) *are satisfied, then*

$$\rho_\lambda(S_J)|_{i,j} = \prod_{p \prec q} \left(1 + \frac{1}{\delta_i(p,q)}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta_j(r,s)}\right) \prod L_k!$$

where the first product is over all pairs $\{p, q\}$ in J which are strict noninversions in T_i, the second product is over all pairs $\{r, s\}$ which are strict *inversions in T_j, and L_k denotes the number of elements of J which lie in the kth row (of either T_i or T_j).*

Proof. When $J = \{k, k+1)$ one has $S_J = I + (k, k+1)$, and Theorem 4.2 is a direct consequence of Definition 3.2. Suppose that $J = \{a, a+1, \dots, a+j-1\}$, and let $\mathscr{T}_J$ denote the set of transpositions $\tau_k = (k, k+1)$ for $k = a, \dots, a+j-2$. Let $M = \rho_\lambda(S_J)$, let $\hat{\tau}_k = \rho_\lambda(\tau_k)$ for $\tau_k \in \mathscr{T}_J$, and let $S_k = \rho_\lambda(I + \tau_k) = I + \hat{\tau}_k$, for $\tau_k \in \mathscr{T}_J$.

Consider the action of S_J on tableaux T_i, $i = 1, \dots, f(\lambda)$. Let $\mathscr{O}^{(\alpha)}$ denote the αth orbit under this action, and let $M^{(\alpha)}$ denote the principal submatrix of M whose rows and columns are determined by $\mathscr{O}^{(\alpha)}$. Let $\hat{\tau}_k^{(\alpha)}$ and $S_k^{(\alpha)}$ denote the submatrices of $\hat{\tau}_k$ and S_k determined by $\mathscr{O}^{(\alpha)}$ in a similar way. By construction, the entries of $\hat{\tau}_k$ are zero outside the blocks $\hat{\tau}_k^{(\alpha)}$. Since S_J is generated by the transpositions in $\mathscr{T}_J$, it follows that M is zero outside the diagonal blocks $M^{(\alpha)}$. This proves statement (1).

The orbits $\mathscr{O}^{(\alpha)}$ are determined by fixing the positions of the elements in $[\mathbf{n}] - J$, and letting the elements of J range freely in the remaining positions, which form a skew subtableau ξ / θ of λ. We now claim:

(a) $M^{(\alpha)} = 0$ if ξ / θ has two cells in the same column;
(b) otherwise, $M^{(\alpha)}$ is a rank-one matrix, with positive entries.

To see this, observe that

$$(I + \tau) S_J = 2 S_J \tag{7}$$

for $\tau \in \mathscr{T}_J$, and hence

$$S_k^{(\alpha)} M^{(\alpha)} = 2 M^{(\alpha)} \tag{8}$$

for $k = a, \dots, a+j-2$. If T_i and T_j are tableaux in $\mathscr{O}^{(\alpha)}$, let $S_k^{(\alpha)}|_i$ and $S_k^{(\alpha)}|_j$ denote the rows of $S_k^{(\alpha)}$ corresponding to T_i and T_j, respectively (and use similar notation for submatrices of M). Suppose that $i < j$ and $T_j = (k, k+1) T_i$. Then $\{k, k+1\}$ is a strict noninversion in T_i which becomes a (strict) inversion in T_j. It follows from Definition 3.2 that

$$S_k^{(\alpha)}|_i = \left(1 + \frac{1}{\delta_i}\right) S_k^{(\alpha)}|_j$$

and hence

$$M^{(\alpha)}|_i = \left(1 + \frac{1}{\delta_i}\right) M^{(\alpha)}|_j,$$

where $\delta_i = \delta_i(k, k+1)$. Given an arbitrary pair of tableaux $T_i, T_j \in \mathscr{O}^{(\alpha)}$, there exists a sequence of transpositions $(k, k+1) \in \mathscr{T}_J$ which transforms T_i into T_j. In fact, one can say a little more. Let T_f and T_l denote the first and last tableaux of $\mathscr{O}^{(\alpha)}$ in last-letter order. (Thus T_f is obtained by reading the columns of ξ/θ in order, and T_l is obtained by reading the rows of ξ/θ in order.) Then one can prove the following:

If T is any tableau in $\mathscr{O}^{(\alpha)}$, it is possible to transform T into T_f by a sequence of transpositions of adjacent letters, each of which is a strict inversion. Similarly, one can transform T into T_l by a sequence of adjacent strict noninversions.[4]

It follows from the observation just made that the rows of $M^{(\alpha)}$ are all multiples of a fixed row, and hence $M^{(\alpha)}$ is a matrix of rank at most one. More precisely, if $T_i \in \mathscr{O}^{(\alpha)}$ is arbitrary, we have

$$M^{(\alpha)}|_i = \prod_{p \prec q} \left(1 + \frac{1}{\delta_i}\right) M^{(\alpha)}|_l,$$

where the product is over all pairs $\{p, q\}$ in J which are strict noninversions in T_i. A similar argument applies to the columns of $M^{(\alpha)}$, and we obtain, for arbitrary T_i and T_j in $\mathscr{O}^{(\alpha)}$,

$$M^{(\alpha)}|_{i,j} = \prod_{p \prec q} \left(1 + \frac{1}{\delta_i}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta_j}\right) M^{(\alpha)}|_{l,f}, \tag{9}$$

where the second product is over the strict inversions in T_j. It remains to determine the constant $C = M^{(\alpha)}|_{l,f}$.

If ξ/θ contains two cells in the same column, then T_f contains two adjacent letters in the same column, say k and $k+1$. But then $S_k^{(\alpha)}|_f = 0$, which implies $M^{(\alpha)}|_f = 0$ *and hence* $M^{(\alpha)}|_{l,f} = 0$ *by* (9).

Next suppose $C \neq 0$, that is, $M^{(\alpha)}$ has rank one. If the representation ρ_λ is restricted to the rows and columns determined by $\mathscr{O}^{(\alpha)}$, one gets a representation $\rho_\lambda^{(\alpha)}$ of S_J. In our notation,

$$\rho_\lambda^{(\alpha)}(S_J) = M^{(\alpha)}.$$

Since S_J transforms to zero in every irreducible representation except the

[4] In fact $\mathscr{O}^{(\alpha)}$ forms a segment in the weak Bruhat order of S_n. This result and a variety of generalizations can be found in [1].

trivial, and since $M^{(\alpha)}$ has rank one, it follows that the trivial representation occurs with multiplicity one in $\rho_\lambda^{(\alpha)}$. Hence

$$\operatorname{trace} M^{(\alpha)} = |J|!.$$

On the other hand, we can compute the trace directly, using (9):

$$\operatorname{trace} M^{(\alpha)} = C \sum_{T_i \in \mathcal{O}^{(\alpha)}} \prod_{p \prec q} \left(1 + \frac{1}{\delta_i(p,q)}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta_i(r,s)}\right).$$

The theorem is proved if we show that $C = \prod L_k!$. But this is exactly what is implied by the following (curious) identity, which we state as a separate result.

LEMMA 4.3. *Let ξ/θ be a skew shape with m cells, containing no two cells in the same column, and let $\mathcal{O}$ denote the set of standard skew tableaux of shape ξ/θ. Let L_k denote the number of cells in the kth row of ξ/θ. Then*

$$\sum_{T \in \mathcal{O}} \prod_{p \prec q} \left(1 + \frac{1}{\delta(p,q)}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta(r,s)}\right) = |\mathcal{O}| = \frac{m!}{\prod L_k!}, \tag{10}$$

where the product is over all strict inversions and strict noninversions in T.

Proof. First extend the product to include all nonstrict noninversions, that is, pairs $p < q$ which lie in the same row of T. Since

$$\prod_{1 \leq a < b \leq L} \left(1 + \frac{1}{b-a}\right) = L! \tag{11}$$

for any $L > 0$, the new terms contribute an additional factor of $\prod L_k!$ to the left hand side of (10). We must therefore show that

$$\sum_{T \in \mathcal{O}} \prod_{p \preccurlyeq q} \left(1 + \frac{1}{\delta(p,q)}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta(r,s)}\right) = m!. \tag{12}$$

Next extend the summation to include nonstandard tableaux as well as standard; in other words, sum over all $m!$ labelings of ξ/θ. The additional terms contribute nothing to the sum, since every nonstandard tableau has an adjacent pair $r > s$ in some row, and for this pair

$$1+\frac{1}{\delta(r,s)}=1+\frac{1}{(-1)}=0.$$

We can now rewrite (12) as

$$\sum_{\sigma\in S_n}\prod_{i<j}\left(1+\frac{1}{x_{\sigma(j)}-x_{\sigma(i)}}\right)=m!, \tag{13}$$

where $x_1, x_2, \ldots, x_m$ are integers such that $x_j - x_i$ equals the axial distance from the ith cell to the jth cell of ξ/θ, numbering cells from bottom left to upper right. For example, one can take x_i to be $c - r$ if i lies in row r and column c. Let $F(x_1, x_2, \ldots, x_m)$ denote the left-hand side of (13). Then

$$\Delta(x_1,\ldots,x_m)F(x_1,\ldots,x_m)=\sum_{\sigma}(\operatorname{sgn}\sigma)\prod_{i<j}(x_{\sigma(j)}-x_{\sigma(i)}+1), \tag{14}$$

where

$$\Delta(x_1,x_2,\ldots,x_m)=\prod_{i<j}(x_j-x_i).$$

The right-hand side of (14) is clearly an alternating function of $x_1, x_2, \ldots, x_m$, and hence is divisible by $\Delta(x_1, x_2, \ldots, x_m)$. It follows that F is a constant, easily seen to be $m!$. This completes the proof of Lemma 4.3. ∎

ILLUSTRATION OF THEOREM 4.2. If $n = 5$, $\lambda = \{3,2\}$, and $J = \{3,4,5\}$, then

$$\rho_\lambda(S_J)=\begin{pmatrix}0&0&0&0&0\\0&4&0&2&\frac{4}{3}\\0&0&0&0&0\\0&\frac{8}{3}&0&\frac{4}{3}&\frac{8}{9}\\0&2&0&1&\frac{2}{3}\end{pmatrix}.$$

If $J = \{2, 3\}$ then

$$\rho_\lambda(S_J) = \begin{pmatrix} \frac{3}{2} & \frac{3}{4} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Note that this matrix is obtained by adding I to the matrix $\rho_\lambda((2,3))$ illustrated earlier.

5. REMARKS

(1) Using (11), one can express the formula for $\rho_\lambda(S_J)|_{i,j}$ in terms of ordinary inversions and noninversions, by suppressing the term $\prod L_k!$. That is,

$$\rho_\lambda(S_J)|_{i,j} = \prod_{p \preccurlyeq q} \left(1 + \frac{1}{\delta_i(p,q)}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta_j(r,s)}\right),$$

where the first product is over (possibly nonstrict) noninversions in T_i.

(2) Using an argument similar to that given in Section 4, one can derive an explicit expression for $\rho_\lambda(S_J^-)$, where

$$S_J^- = \sum_{\sigma \in S_J} (\operatorname{sgn} \sigma)\sigma.$$

The nonzero matrix entries occur in the same positions as $\rho_\lambda(S_J)$, and are given by

$$\rho_\lambda(S_J^-)|_{i,j} = (\operatorname{sgn} \sigma) \prod_{p \prec q} \left(1 + \frac{1}{\delta_i(p,q)}\right) \prod_{r \succ s} \left(1 + \frac{1}{\delta_j(r,s)}\right) \prod L_k!,$$

where σ is the permutation carrying T_i into T_j.

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